Study of an Orbiting Tethered Dumbbell System

Having Positive Orbital Energy

Contract NAS8-35497

Addendum to the Final Report

For the period 16 August 1983 through 28 February 1985

Principal Investigator

Dr. Enrico Lorenzini

February 1985

Prepared for
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama 35812

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CONTENTS

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The Smithsonian Astrophysical Observatory
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Abstract

For very long tethered systems the sum of the kinetic and potential energy can be positive. The system remains in a circular orbit as long as the masses remain vertically aligned. The system is unstable without constant control of the alignment. If the upper mass rotates forward in the direction of the orbital motion, the system escapes out of orbit. If the upper mass rotates backward, the system falls out of orbit and the lower mass impacts the body around which the system is orbiting.

Acknowledgement

This report was prepared by Mr. David A. Arnold as an addendum to "Analytical Investigations of the Dynamics of Tethered Constellations in Earth Orbit," (E. Lorenzini, D.A. Arnold, and M.D. Grossi), Final Report NAS8-35497, December 1984. The author wishes to thank Enrico Lorenzini and Gordon Gullahorn of the Smithsonian Astrophysical Observatory, and Joseph A. Carroll of the Energy Science Laboratories Inc. for reviewing and confirming the calculations, and Dr. Georg von Tiesenhausen of Marshall Space Flight Center for initiating this investigation.

1.0 INTRODUCTION

Under NASA Contract NAS8-33691 (Orbital transfer and release of tethered payloads, Colombo, March 1983) a study was performed of the work done by the reel motor in retrieving a system consisting of two equal masses connected by a massless tether. Table 4-1 of that report shows the altitudes of the masses, the center of mass, and the orbital center before retrieval, and the final altitude and increase in total orbital energy (ΔTE) after retrieval. For the beginning entries the change in orbital energy ΔTE is proportional to the square of the tether length as calculated by equation 20 and shown in the column headed w. The last entry shows only about a 10 percent increase in energy over the previous entry even though the tether length increases by a factor of two. The attempts to understand this discrepancy in the last entry led to the discovery of a dumbbell system having positive total energy but able to exist in a bound circular orbit as long as the alignment is carefully maintained.

2.0 ORBITAL ENERGY VS. TETHER LENGTH BEFORE AND AFTER RETRIEVAL

Assuming that no thrusters are used, the orbital angular momentum of a tethered system must remain constant during retrieval. Assuming that the retrieval is done very slowly so that no eccentricity is introduced in the orbit, the final orbit radius R of a system of masses connected by a massless tether after retrieval is given by:

$$R = [(\Sigma r_i^2 m_i)^2 \Sigma m_i/r_i^2]/[(\Sigma m_i)^2 \Sigma r_i m_i] \tag{1}$$

where r_i are the orbital radii before retrieval and m_i are the masses of the bodies. The orbital angular velocity ω before retrieval is

$$\omega = [(\Sigma GMm_i/r_i^2)/(\Sigma m_i r_i)]^{1/2} \tag{2}$$

where GM is the gravitational constant of the earth. The angular velocity Ω after retrieval is

$$\Omega = [GM/R^3]^{1/2} \tag{3}$$

Using these equations the sum of the kinetic and potential energy before and after retrieval can be calculated in order to see the change in total orbital energy during the retrieval process. This calculation of the change in energy is based solely on the assumption of conservation of angular momentum and circular orbits before and after retrieval.

A set of calculations has been done for the case of a 10 metric ton mass at 200 km altitude connected by a massless tether to another 10 metric ton mass at various higher altitudes. The earth's radius is taken to be 6378 km. Table 1a lists vs. tether length the energy E_1 of the lower mass, the energy E_2 of the upper mass, the total energy E_{TOTAL} of the system in the deployed state and the energy E_f of a 20 metric ton mass having the same orbital angular momentum in a circular orbit (i.e., after retrieval to a circular orbit). Table 1b lists vs. tether length, the altitude after retrieval H_f (corresponding to E_f), the altitude of the center of mass H_{cm}, the orbital center H_{oc}, and center of energy H_E. Appendix 1 gives a detailed calculation for the first entry in the table showing all forces, potential and kinetic energies, and angular momentum for each mass. The values in Table 1 show various transition points. These transition points have been obtained by iterative interpolation of the results of a small computer program written to calculate the various quantities.

Table 1a

ℓ (km)	E_1 (ergs) x10^{19}	E_2 (ergs) x10^{19}	E_{TOTAL} (ergs) x10^{19}	E_f (ergs) x10^{19}	ΔE (ergs) x10^{18}
12,800	-.5203	+.5374	.0171	-.0916	-1.0866
12,500	-.5190	+.5222	.0031	-.0954	-.9852
12,450	-.5188	+.5196	.0008	-.0960	-.9685
12,432.7728325	-.5187	+.5187	0.	-.0963	-.9627
12,400	-.5186	+.5171	-.0015	-.0967	-.9518
12,300	-.5182	+.5120	-.0062	-.0981	-.9186
12,000	-.5168	+.4966	-.0202	-.1022	-.8203
11,000	-.5119	+.4449	-.0670	-.1178	-.5081
10,000	-.5063	+.3923	-.1140	-.1364	-.2236
9,500	-.5033	+.3656	-.1377	-.1470	-.0934
9,200	-.5013	+.3494	-.1519	-.1539	-.0198
9,150	-.5010	+.3467	-.1543	-.1551	-.0078
9117.085092613	-.50076	+.3449	-.1558	-.1558	0.
9,100	-.5006	+.3440	-.1566	-.1562	+.0040
9,000	-.4999	+.3385	-.1614	-.1587	+.0274
8,000	-.4926	+.2835	-.2091	-.1853	+.2377
7,000	-.4838	+.2266	-.2572	-.2174	+.3991
6,400	-.4779	+.1915	-.2864	-.2395	+.4688
6,000	-.4734	+.1675	-.3059	-.2556	+.5028
5520.83	-.4677	+.1383	-.3295	-.2765	+.5295
5,000	-.4607	+.1055	-.3552	-.3012	+.5403
4970.9828	-.4603	+.1037	-.3566	-.3026	+.5403
4,000	-.4448	+.0398	-.4051	-.3545	+.5060
3429.63571136	-.4339	0.	-.4339	-.3884	+.4544
3,200	-.4289	-.0166	-.4455	-.4027	+.4276
2969.100184	-.4236	-.0336	-.4572	-.4175	+.3974
2,500	-.4118	-.0693	-.4811	-.4484	+.3278
1,600	-.3835	-.1436	-.5271	-.5094	+.1765
800	-.3496	-.2178	-.5674	-.5619	+.0551
400	-.3283	-.2587	-.5870	-.5855	+.0153
200	-.3162	-.2804	-.5966	-.5962	+.0040
100	-.3097	-.2916	-.6013	-.6012	+.0010

Table 1b

ℓ (km)	H_f (km)	H_{cm} (km)	H_{oc} (km)	H_E (km)
12,800	37,157.64333	6600.	3645.51873	
12,500	35,409.79527	6450.	3595.87502	
12,450	35,123.72395	6425.	3587.53457	
12,432.7728325	35,025.50417	6416.38642	3584.65646	∞
12,400	34,839.13772	6400.	3579.17481	2,602,764.988
12,300	34,274.40392	6350.	3562.39684	637,516.5735
12,000	32,615.41283	6200.	3511.58661	191,043.0744
11,000	27,456.84164	5700.	3336.73648	53,119.00197
10,000	22,844.41838	5200.	3152.53361	28,572.51997
9,500	20,732.81257	4950.	3056.49402	22,572.82820
9,200	19,525.79670	4800.	2997.49551	19,862.67261
9,150	19,328.91911	4775.	2987.55784	19,459.05118
9117.085092613	19,199.97869	4758.542546	2980.99926	19,199.97869
9,100	19,133.25708	4750.	2977.58967	19,067.51553
9,000	18,745.56381	4700.	2957.56091	18,318.58861
8,000	15,128.22448	4200.	2750.06896	12,683.45823
7,000	11,961.06611	3700.	2527.87689	9115.90936
6,400	10,264.18951	3400.	2386.39203	7539.87911
6,000	9213.76598	3200.	2288.23953	6651.23130
5520.83	8037.61137	2960.415	2166.24284	5720.83139
5,000	6857.47122	2700.	2027.66910	4844.17580
4970.9828	6794.65979	2685.49140	2019.75326	4798.89170
4,000	4865.54904	2200.	1741.69697	3461.30790
3429.63571136	3883.53370	1914.81786	1565.06573	2808.88909
3,200	3518.65195	1800.	1490.80999	2568.88570
2969.100184	3169.100184	1684.55009	1414.18794	2339.33329
2,500	2511.78707	1450.	1252.05002	1906.21554
1,600	1446.20412	1000.	913.59161	1184.21745
800	715.55513	600.	577.09583	646.62239
400	429.56898	400.	394.10026	411.85446
200	307.49130	300.	298.50266	302.99827
100	251.88620	250.	249.62282	250.75459

The most significant transition point in Table 1 occurs at a tether length of 12,432.7728325 km. At this point the total orbital energy of the system is zero. For longer tether lengths, the total energy is positive meaning that the system has sufficient energy to escape from orbit.

At a tether length of 9117.085092613 km, the energies before and after retrieval are equal so that the retrieval energy is zero. Since the tension in the deployed state is quite high, the reel motor must do a lot of work at the beginning of retrieval. Since the tether cannot push, it is difficult to see how the tether can do work on the reel motor during retrieval unless some type of yo-yo maneuver could be devised such that there is a net loss of energy per cycle. If the tether is librating the tension is greater when the tether angular momentum is parallel to the orbital angular momentum and less when it is antiparallel. No simulations have been done to see whether it is in fact possible to retrieve a tether of this length. The retrieval may be intrinsically unstable beyond a certain tether length, but this point has not been investigated.

The maximum retrieval energy occurs at a length of 4970.9828 km. Below this length the retrieval energy keeps increasing with tether length. This point must still be in a region where the tether would have to do work on the reel motor in order to achieve a stable retrieval.

The energy of the upper mass becomes positive at a tether length of 3429.63571136 km. This means that if the upper mass were released from the upper end of the tether it would escape from orbit.

At a tether length of 2969.100191 km the final altitude after retrieval is equal to the altitude of the upper mass before retrieval.

Figure 1 shows a plot of the retrieval energy ΔE vs tether length up to the point where the retrieval energy goes negative at 9117.085092613 km. For short tether lengths the retrieval energy is proportional to the square of the tether length. Past about 1600 km the curve starts to show significant departures from the ℓ^2 law reaching a maximum value just before 5000 km. The curve goes negative just past 9000 km.

For equal masses, the transition points seen in Table 1 are a function only of the ratio of the orbital radii of the masses in the deployed state.

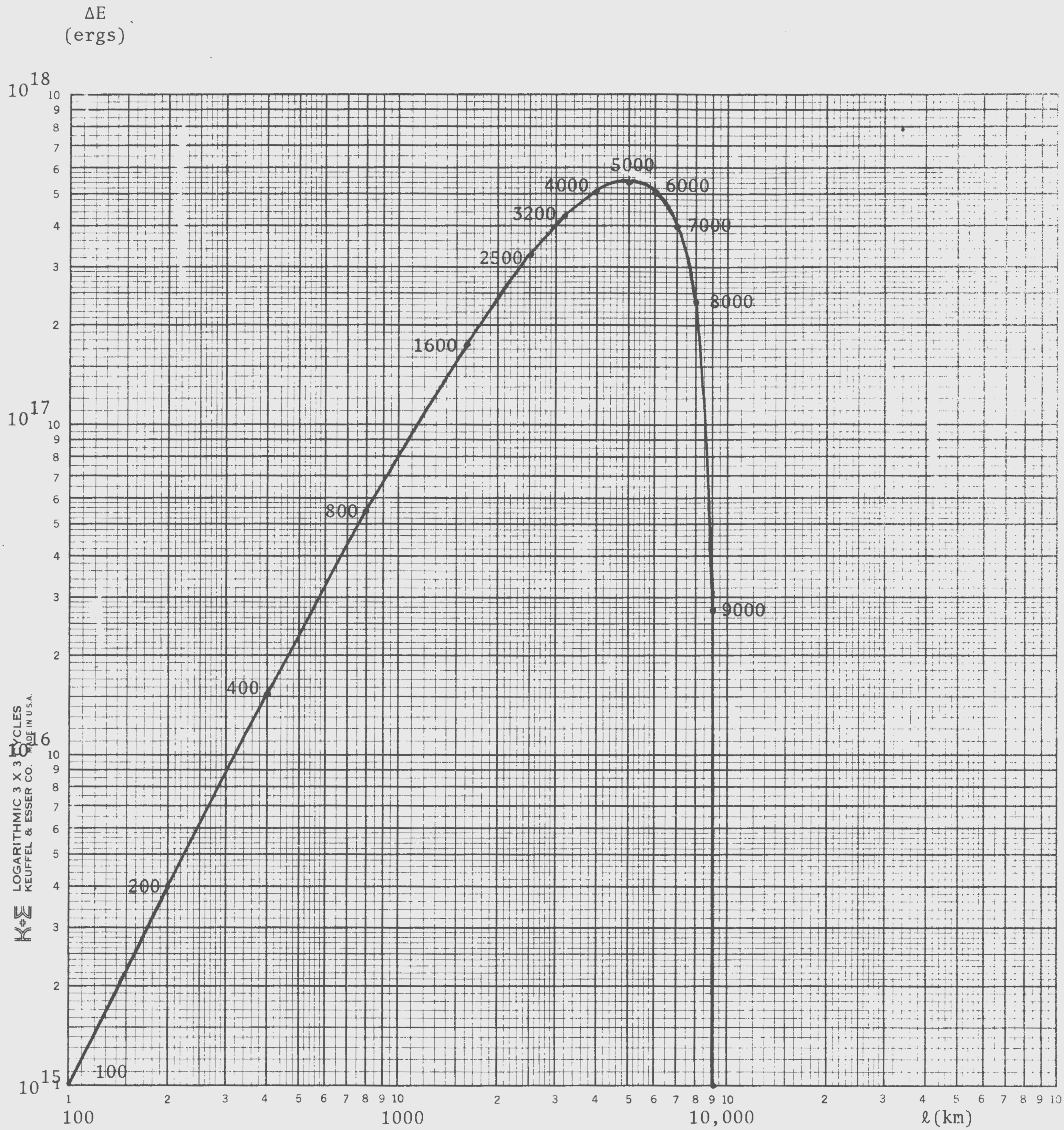


Figure 1

Retrieval energy vs. tether length

Table 2 lists the ratio of the orbital radii for each of the transition points of Table 1. The orbital radius of the upper mass is r_2 and that of the lower mass is r_1.

Table 2

r_2/r_1	Type of Transition
2.890053638	Total system energy equal to zero
2.385996517	Retrieval energy equal to zero
1.7556982	Maximum retrieval energy
1.521379707	Energy of upper mass equal to zero
1.451368226	Altitude after retrieval equal to that of the upper mass before retrieval

The ratio r_1/r_2 for the point where the total energy is zero is .3460143. Appendix 2 shows a derivation devised by Gordon Gullahorn confirming the possibility of positive total energy and a computer calculation showing the point where the total energy goes positive.

3.0 COMPUTER SIMULATION OF POSITIVE ENERGY DUMBBELL IN A CIRCULAR ORBIT

A short computer simulation has been done to verify that a system with positive orbital energy can exist in a bound circular orbit. The lower mass is at 200 km altitude, the tether is 12,800 km long and the upper mass is at 13,000 km. Each mass is 10 metric tons. The wire joining them has a stiffness of 22.34 dynes/cm and is modelled as a visco-elastic connection with critical damping. The tension is $.6608 \times 10^{10}$ dynes. The mass of the wire is not modelled in the simulation. The orbital angular velocity is .000629128 radians/second. This is the case discussed in Appendix 1 and shown as the first item in Table 1. The simulation was run for 1000 seconds of orbital time. The orbital radius remained at 6578 km for the lower mass and 19,378 km for the upper mass, and the wire tension was constant. No out-of-plane displacement developed. The upper mass developed a slight displacement of about $.2 \times 10^{-5}$ cm in the forward direction by the end of the run. Otherwise, the system remained in a perfect circular orbit.

4.0 COMPUTER SIMULATIONS OF INSTABILITY MODES

4.1 Initial 1° Forward Rotation

The simulation described in the last section shows a very small in-plane displacement developing. In order to speed up the development of any potential instabilities the upper mass was started with a 1° displacement forward in the direction of the orbital motion. The in-plane displacement of the upper mass relative to the center of the system has been plotted vs time. At t = 0, the displacement was $.11 \times 10^8$ cm forward which is 1°. The upper mass swung back through the vertical position at about 1300 seconds of orbital time and continued to the rear reaching a maximum of about $.42 \times 10^7$ cm (about .38 degrees) at 2000 seconds. It swung forward again crossing the vertical at about 2600 seconds and then continued forward tumbling end over end. The system had rotated 90° forward at 12,200 seconds and reached 180° (upside down) at 16300 seconds. The run was stopped at 20,000 seconds. At that time the lower mass had reached a distance of 64,036 km from the center of the earth and the upper mass was at 63,752 km. At the beginning of the run the orbital energy of the system was $.17074 \times 10^{18}$ ergs. By the end of the run the energy had increased to $.937 \times 10^{18}$ ergs due to the contraction of the wire as the gravity gradient force decreased. The work done by the wire on the system was obtained by integrating the tension vs the change in length. This amounted to $.766 \times 10^{18}$ ergs as the wire contracted from 12,800 km to 10,315 km. The tension decreased from $.6608 \times 10^{10}$ to $.1045 \times 10^{10}$ dynes. The kinetic energy of the upper mass at t = 0 is $.74 \times 10^{19}$ ergs which is an order of magnitude larger than the work done by the contraction of the tether. The wire stiffness was 22.34 dynes/cm.

4.2 Instability From Numerical Roundoff

As a result of numerical roundoff, the initial conditions for the simulation are not exactly in equilibrium. The system should eventually go unstable without any deliberate introduction of perturbations. A long integration has been done to see how the instability develops from a condition of almost perfect equilibrium. In order to reduce stretching of the wire the stiffness was increased to 1723 dynes/cm. The run proceeded smoothly for about 40,000 seconds of orbital time and then encountered a region of very slow numerical integration and small stepsizes. The run was terminated at 41,500 seconds. A plot of the in-plane displacement showed that the upper mass had first rotated forward very slightly by $.2 \times 10^{-7}$ cm at 200 seconds and then started falling to the rear with the displacement increasing geometrically with time. Table 3 shows the displacement vs time. The displacement increases about an order of magnitude every 3200 seconds.

Table 3

Time (sec)	In-Plane Displacement (cm)
1100	.00001
3100	.00010
6100	.00101
9300	.0102
12,500	.101
15,700	1.01
18,900	10.0
22,200	107.
25,400	1063.
28,600	10,558.
31,800	104,827.
35,000	1,042,811.
38,200	10,582,531.
41,200	106,740,452.

A plot of the distance of each mass from the center of the earth shows that the altitude of the system decreases as the upper mass rotates to the rear against the direction of motion. The initial geocentric distance of the lower mass was 6578 km (200 km altitude). This decreased to 6378 km (the radius of the earth) at around 37,800 seconds. Since the integration program treats the earth as a point source, the integration continued until the lower mass was at a distance of 162 km from the center of the earth. This explains the slow integration since the lower mass was approaching the center of force. Because of the strong attraction of the force center, the tether started stretching, and the orbital energy decreased because of the work done on the tether. Apparently, if the upper mass rotates to the rear, the system falls out of orbit instead of escaping as it did for forward rotation of the upper mass.

4.3 Initial .1° Backward Rotation

This simulation has been done with the upper mass initially rotated .1° to the rear against the direction of motion. The wire stiffness was 280 dynes/cm. The upper mass first rotated forward passing through the vertical reaching a maximum forward displacement of about .06° at about 2150 seconds. It then rotated backward, passing the vertical at around 3000 seconds and continuing backward with continually increasing speed. The run was terminated at 10,100 seconds because the integration was becoming difficult. The altitude of the lower mass reached the radius of the earth at about 6700 seconds and the lower mass was 1791 km from the force center at the end of the run. The wire was stretching and the orbital energy decreasing sharply at the end.

4.4 Initial .1° Forward Rotation

In this run the initial displacement of the upper mass is .1° forward and the wire stiffness is 73.88 dynes/cm. The integration proceeded smoothly and was allowed to continue to 50,000 seconds of orbital time. The upper mass initially rotated to the rear, passing the vertical at about 1300 seconds and reaching a maximum displacement of about .053 degrees to the rear at 2100 seconds. The wire then rotated forward passing the vertical at 2900 seconds and continuing forward with increasing speed. Table 4 gives the time at which the system passes various orientations. The angle of rotation is measured in a coordinate system rotating with the orbit of the center of mass. Also included is the energy of the system, the tension, and geocentric distance of each mass for each time.

Table 4

Angle (Deg)	Time (Sec)	Energy (Ergs)	Tension (Dynes)	R_1 (km)	R_2 (km)
.1	0	$.1710 \times 10^{18}$	$.6608 \times 10^{10}$	6578	19,378
0.	1300	.1716	.6598	6580	19,379
-.05	2100	.1720	.6595	6585	19,383
0.	2900	.1725	.6587	6593	19,391
90.	19,400	.4339	.0395	44,896	44,897
180	26,150	.4333	.0487	73,000	61,000
270	32,250	.4336	.0449	86,000	86,000
360	38,250	.4335	.0466	97,000	109,000
450	44,100	.4336	.0455	119,000	119,000
540	49,950	.4335	.0462	140,000	129,000

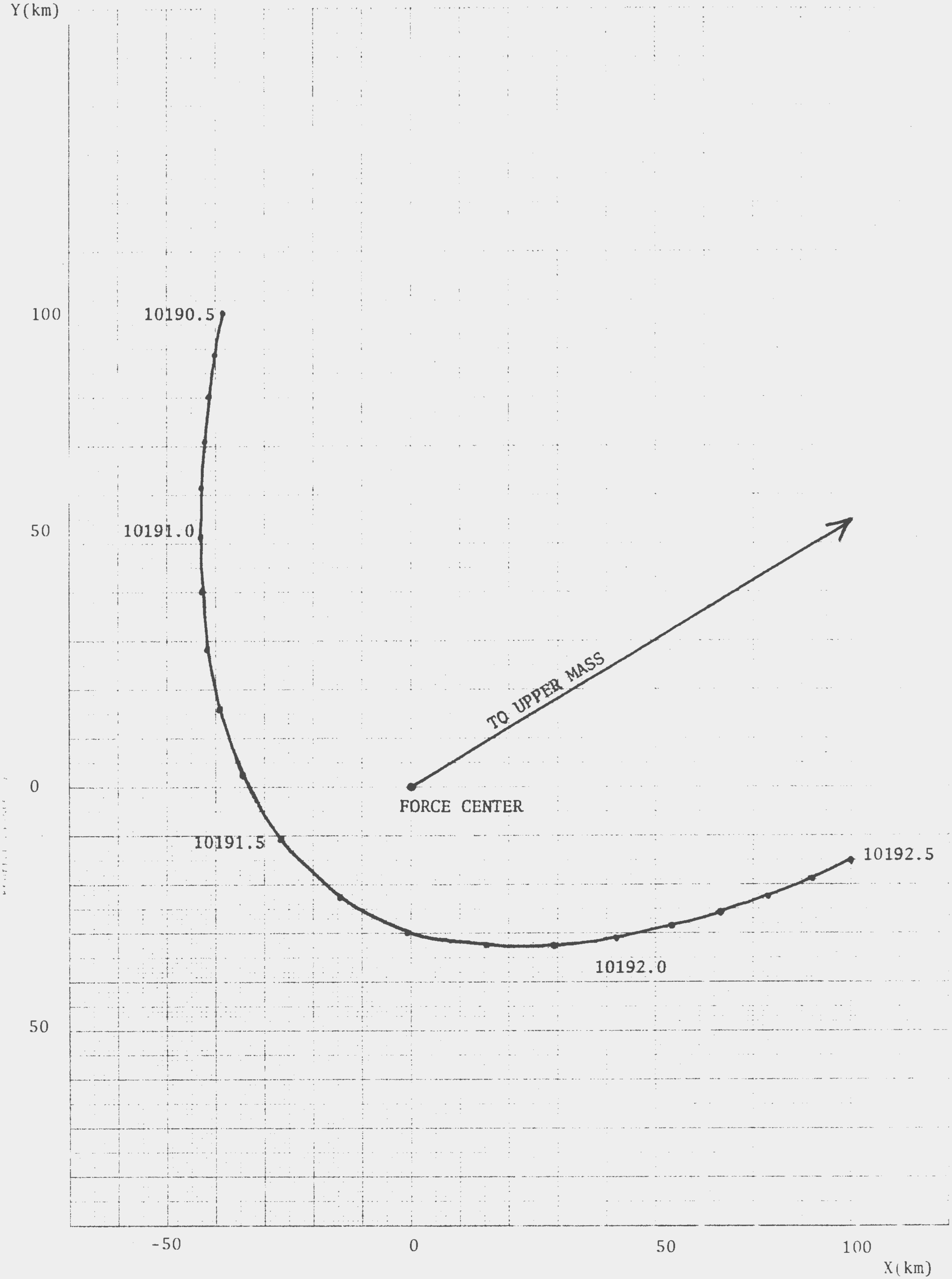
4.5 Continuation Of .1° Backward Rotation

This run is a continuation of the run described in Section 4.3 in an attempt to carry the integration as far as feasible without running into prohibitively slow integration. Output was started at t = 9900 and continued to 10192.7 seconds, decreasing the output interval from 1.00, to .10, and finally to .01 seconds. The lower mass came within 26.7 km of the force center at 10191.6 seconds. Figure 2 shows a plot of the path of the lower mass at .1 second intervals from 10190.5 seconds to 10192.5 seconds. The arrow shows the direction to the upper mass at 10191.5 seconds. The unsymmetrical path may be due to the damping included in the model of the tether.

4.6 Initial .1° Displacement Out-Of-Plane

In this run, the tether is given an initial displacement of .1° in the out-of-plane direction. The tether is aligned with the inertial x-axis, the orbital velocity is toward the +y-axis and the displacement of the upper mass is toward the +z-axis. The upper mass executes a sinusoidal oscillation in the out-of-plane direction with a period of 4350 seconds. The amplitude decreased to about .08° toward the end of the run which was at 17,000 seconds. The orbital period is 9988 seconds. Half the orbital period is 4994 seconds which differs by about 15 percent from the out-of-plane period. For short tethers, the out-of-plane period is half the orbital period. The upper mass develops an in-plane displacement to the rear. The lower mass reaches the earth's radius at about 13,800 seconds and was 3183 km from the force center at the end of the run.

Figure 2. Trajectory of the lower mass near the center of force.



4.7 Initial 1° Forward Rotation With A Stiff Wire

This run is a repeat of the run in Section 4.1 with the wire stiffness set to 42,957 dynes/cm and a 1° initial displacement of the upper mass in the forward direction. The purpose of using a high stiffness is to minimize energy exchange between the tether and the orbit. The upper mass rotates to the rear, crossing the vertical at about 1250 seconds, reaching a maximum displacement of .66 degrees at around 2150 seconds, and then swings forward again crossing the vertical at about 3050 seconds. The mass then continues forward for the rest of the run. The system rotated almost a quarter of a turn (about 89°) by the end of the run at 20,000 seconds. The final distance of the lower mass from the center of the earth was 50,300 km and that of the upper mass was 50,500 km. The orbital energy increased from $.17074 \times 10^{18}$ ergs to $.17124 \times 10^{18}$ ergs during the run and the tension decreased from $.6608 \times 10^{10}$ dynes to $.252 \times 10^{9}$ dynes. The work done by the tether was only $.505 \times 10^{15}$ ergs. Figure 3 shows a plot of the in-plane displacement (meters) of the upper mass with respect to the center of the system vs time. The sign convention used in the plot is that positive displacements are to the rear. Figure 4 shows the radial component (meters) of the upper mass with respect to the lower mass as a function of time. Figure 5 shows the tension vs time. Figure 6 shows a side view of the system with successive configurations displaced to the right.

This run was repeated for the first 9000 seconds without damping in the tether model. The integration was slower but the numerical results were almost the same.

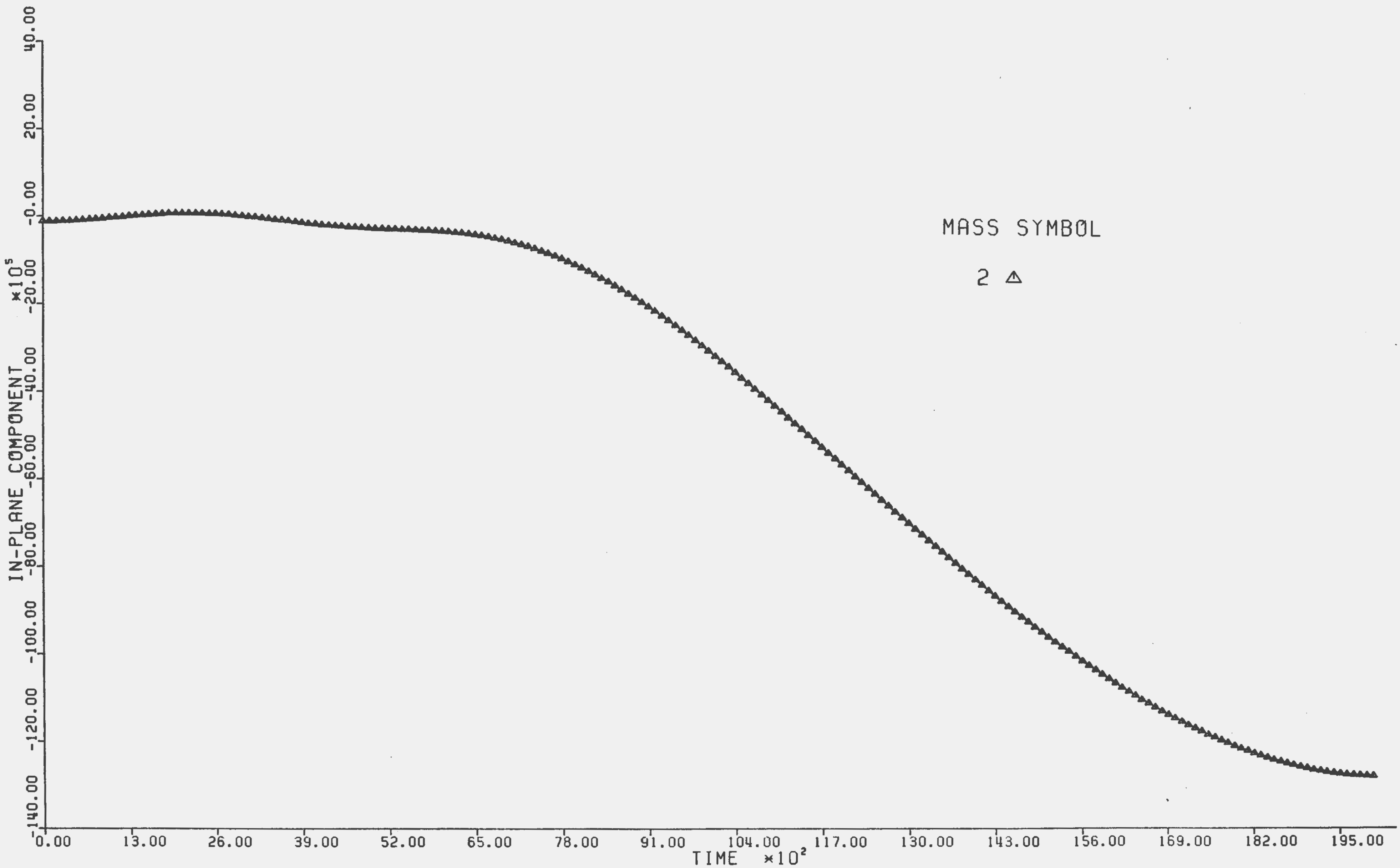
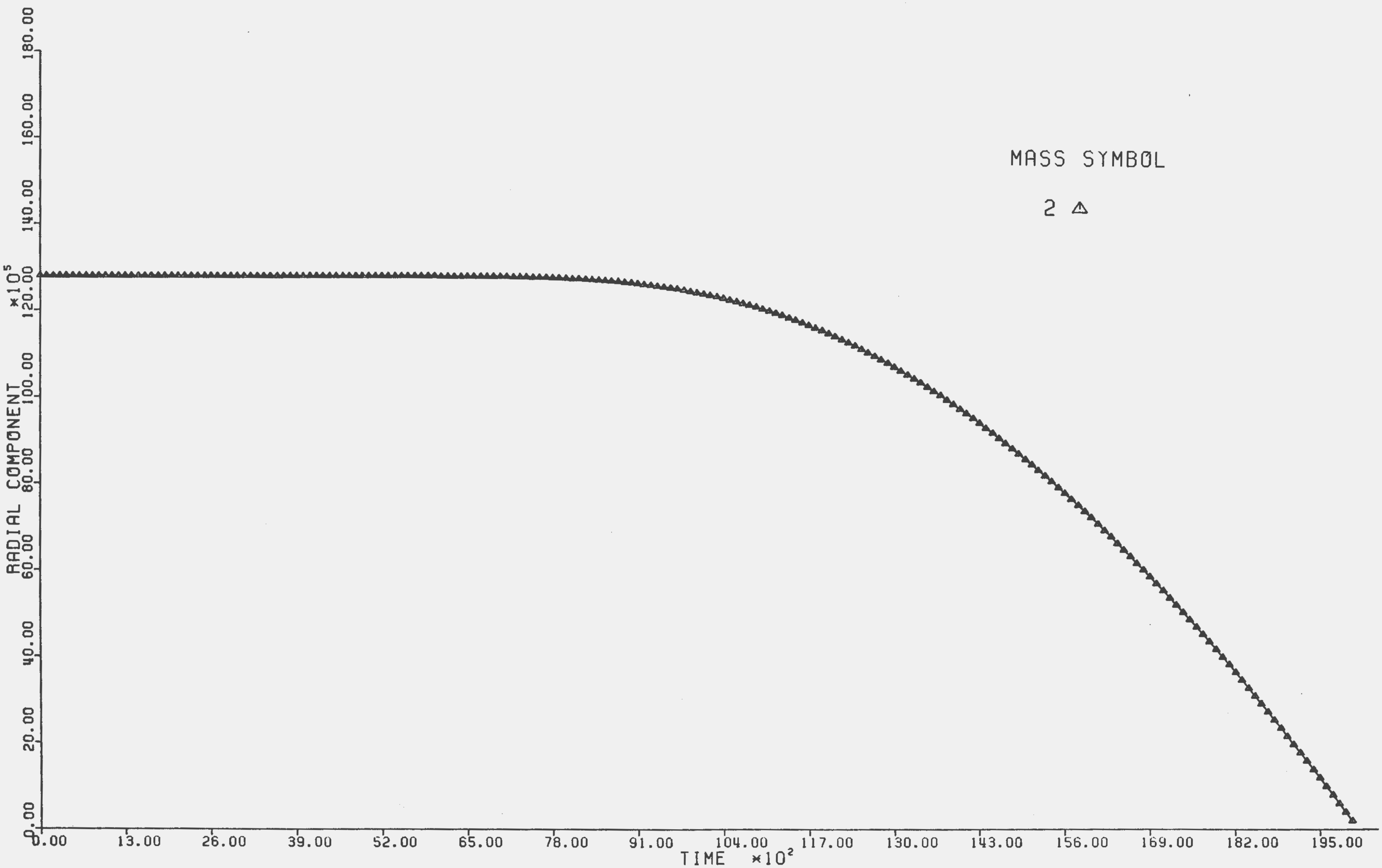

MASS SYMBOL
2 △
IN-PLANE COMPONENT ×10⁵
40.00
20.00
-0.00
-20.00
-40.00
-60.00
-80.00
-100.00
-120.00
-140.00
0.00
13.00
26.00
39.00
52.00
65.00
78.00
91.00
104.00
117.00
130.00
143.00
156.00
169.00
182.00
195.00
TIME ×10²

Figure 3.


MASS SYMBOL
2 △
RADIAL COMPONENT ×10⁵
TIME ×10²
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
160.00
180.00
13.00
26.00
39.00
52.00
65.00
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104.00
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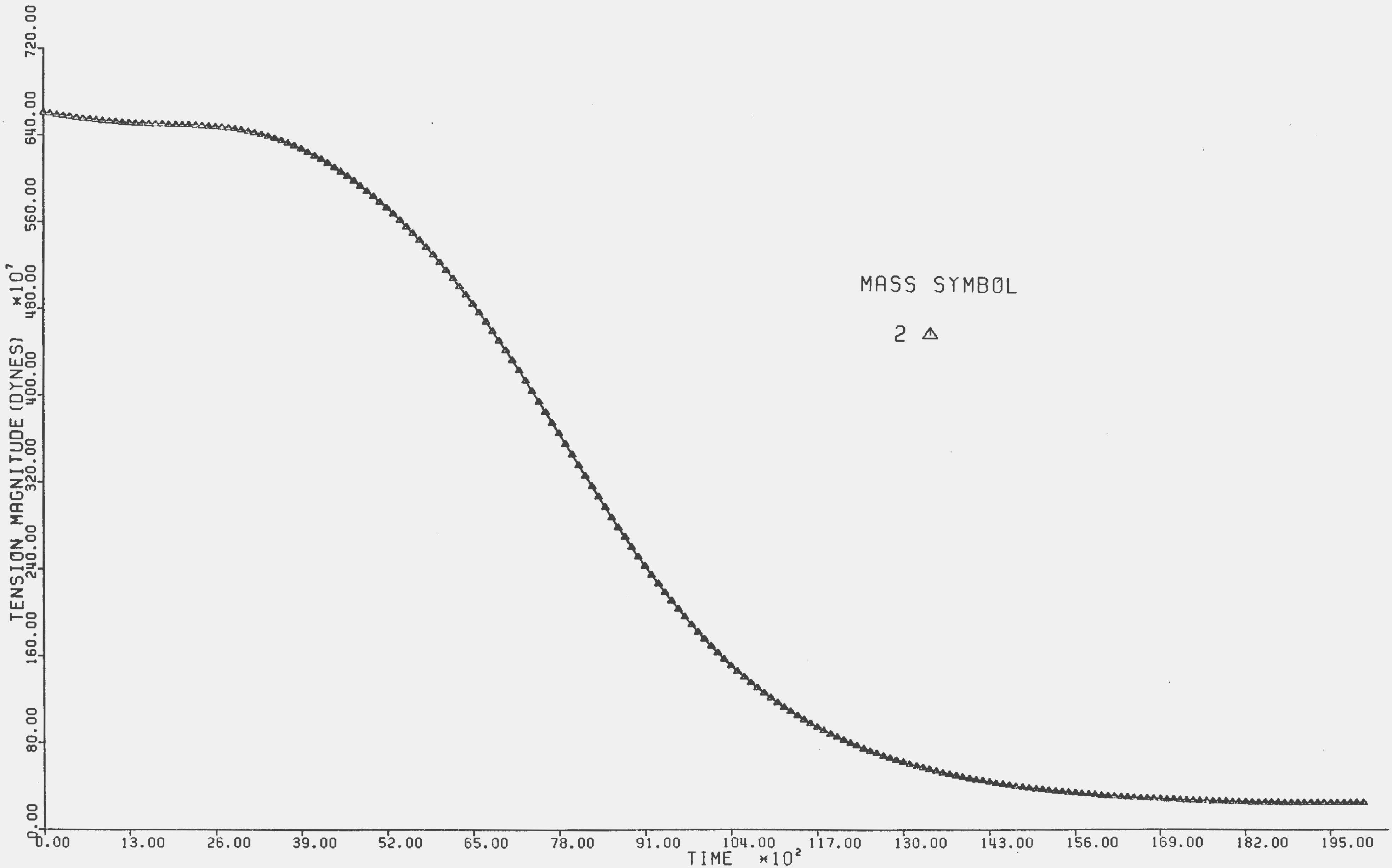
130.00
143.00
156.00
169.00
182.00
195.00

Figure 4.



MASS SYMBOL
2 △
TENSION MAGNITUDE (DYNES) ×10⁷
TIME ×10²
0.00
80.00
160.00
240.00
320.00
400.00
480.00
560.00
640.00
720.00
0.00
13.00
26.00
39.00
52.00
65.00
78.00
91.00
104.00
117.00
130.00
143.00
156.00
169.00
182.00
195.00

Figure 5.

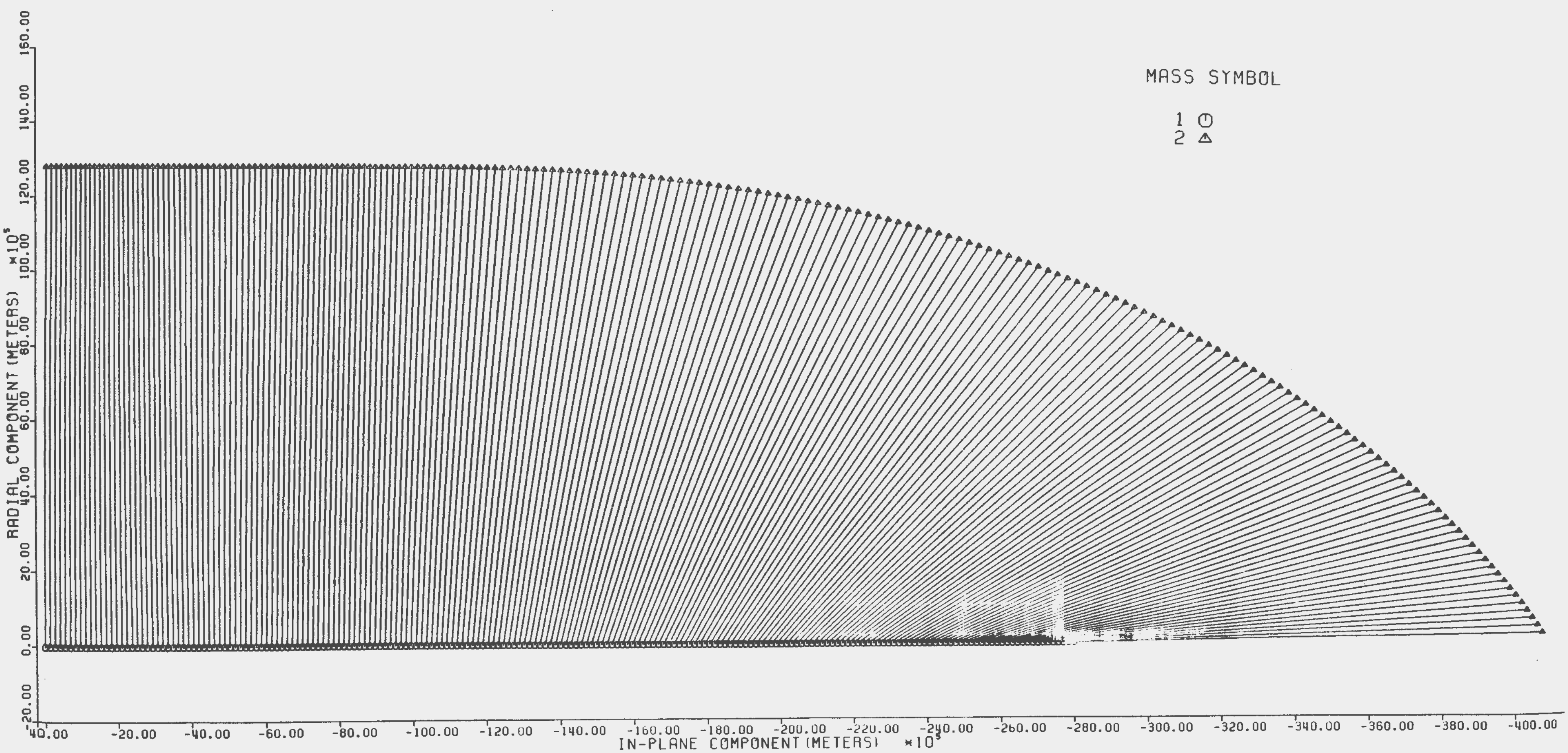

MASS SYMBOL
1 ⦶
2 △
RADIAL COMPONENT (METERS) ×10⁵
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
-20.00
IN-PLANE COMPONENT (METERS) ×10⁵
-20.00
-40.00
-60.00
-80.00
-100.00
-120.00
-140.00
-160.00
-180.00
-200.00
-220.00
-240.00
-260.00
-280.00
-300.00
-320.00
-340.00
-360.00
-380.00
-400.00

Figure 6.

4.8 Initial .1° Backward Rotation With A Stiff Wire

This case is the same as that of Section 4.3 and continued in Section 4.5, but with the stiffness increased to 42,957 dynes/cm. The purpose of the high stiffness is to minimize the exchange of energy between the tether and the orbit. In Section 4.5, the lower mass reached the center of force at 10191.6 seconds. For this run the geocentric distance at 10200 seconds is 3730 km and the energy is $.1678 \times 10^{18}$ ergs (2 percent less than the initial $.171 \times 10^{18}$ ergs). The run in Section 4.5 reached 3913 km at 9700 seconds and the energy had dropped to $-.08 \times 10^{18}$ ergs. The present run was terminated at 10,600 seconds with the lower mass at 1506 km. In Section 4.5, this altitude was reached at 10125 seconds (about 67 seconds before closest approach to the force center). The higher stiffness delayed the decay by about 475 seconds by reducing energy absorption by the tether.

5.0 LONG TETHER WITH A HEAVY MASS AT THE LOWER END

The instability modes described in Section 4.0 are not solely a function of tether length. This section describes a run with the same orbital parameters (12,800 km tether, lower mass at 200 km, upper mass at 13,000 km) except that the lower end is given an extremely large mass (10^9 metric tons) while the upper mass remains at 10 tons. The upper mass is given an initial 1° rotation in the forward direction. The tension is $.26 \times 10^{11}$ dynes, and the orbital period is 5310 seconds. The simulation was run for 12,000 seconds of orbital time. The upper mass executed a stable sinusoidal oscillation having a period of about 7500 seconds. This is over twice as long as the period of 3065 seconds for short tethers.

6.0 INSTABILITY OF NEGATIVE ENERGY STATES

In Table 1, the energy of the tethered dumbbell becomes positive for a tether length of 12,432.77 km. In order to see if negative energy states are stable, a run has been done for a tether length of 12,000 km. The tether was given an initial in-plane displacement of .1 degree in the forward direction. The run exhibited the same instability as the runs for a 12,800 km tether. The initial displacement was reduced to .001 and finally to .000001 degrees with the same results. The next transition point is at 9117.08 km where the retrieval energy is zero. A run was done with a tether length of 9000 km and an initial displacement of .000001 degrees. This run was also unstable. The next transition point is at a length of 4970.98 km when the maximum retrieval energy occurs. A run done at 4800 km appeared to be stable during the first 20,000 seconds. The initial displacement was .000001 degrees. Runs at 5200 km and 5600 km also appeared to be stable. For a tether length of 6000 km, the in-plane angle was stable, but the altitude was not. The altitude kept increasing at an accelerating pace. For a length of 7000 km, both the angle and altitude were unstable. The in-plane angle which started in the forward direction oscillated to the rear twice before continuing forward at an accelerated pace.

The instability seems to set in somewhere above the point of maximum retrieval energy. The column in Table 1b giving the center of energy H_E may provide an explanation. The center of energy is defined as the radius of the orbit of a single particle having the same mass and energy as the tethered system in the deployed state. In Table 1b the center of energy is at the altitude of the upper mass for a tether length of 5520.83 km. For the case run with a 5600 km tether, the center of energy is at 5865 km which is just above the upper mass at 5800 km. This is the highest run that appears to be stable (at least for very small in-plane oscillations). All the runs with longer tethers were unstable.

7.0 SUMMARY OF RESULTS

In very long tethers it is possible for the system to have positive orbital energy and yet be in a bound orbit. However, the system is unstable with respect to in-plane librations. If the angular momentum of the libration is parallel to the orbital angular momentum the system escapes from orbit, and if it is anti-parallel, the system falls out of orbit. Neglecting tether elasticity both processes take place at constant orbital energy. The system is stable with respect to out-of-plane librations, but the coupling between the in-plane and out-of-plane motions can destabilize the in-plane angle. The instability of the in-plane angle may be associated with the point where the center of energy goes outside the system. The elasticity of the tether increases the orbital energy during escape and decreases the orbital energy during orbital collapse.

8.0 APPENDIX I: COMPLETE CALCULATION OF THE ENERGY, FORCES, AND ANGULAR MOMENTUM FOR A 12,800 KM TETHERED DUMBBELL

A mass of 10 metric tons at 200 km altitude is connected by a 12,800 km tether to another mass of 10 metric tons at 13,000 km. The system has the same orbital angular momentum as a mass of 20 metric tons at an altitude of 37,157.64333 km. The total energy of the deployed system is positive even though it is in a bound orbit. The radial acceleration of the system is zero with an angular velocity of $.6291280459 \times 10^{-3}$ radians/second. The total centrifugal force on the system with $R_e = 6378$ km is

$$m(r_1 + r_2)\omega^2$$

$$= 10 \times 10^6 \ (6578 \times 10^5 + 19378 \times 10^5) \times (.6291280459 \times 10^{-3})^2$$

$$= 1.027343926 \times 10^{10} \text{ dynes.}$$

The total gravitational force on the system is

$$GMm(1/r_1^2 + 1/r_2^2)$$

$$= 3.986013 \times 10^{20} \times (10 \times 10^6) \times [(6578 \times 10^5)^{-2} + (19378 \times 10^5)^{-2}]$$

$$= 1.027343926 \times 10^{10} \text{ dynes.}$$

the system is therefore in equilibrium in a bound orbit.

The kinetic energy of each mass is

$$m_1 r_1^2\omega^2/2 = (10 \times 10^6) \times (6578 \times 10^5)^2 \times (.6291280459 \times 10^{-3})^2/2$$

$$= .8563195015 \times 10^{18} \text{ ergs}$$

$$m_2 r_2^2\omega^2/2 = (10 \times 10^6) \times (19378 \times 10^5)^2 \times (.6291280459 \times 10^{-3})^2/2$$

$$= .7431320625 \times 10^{19} \text{ ergs}$$

The potential energy of each mass is

$-GMm_1/r_1 = -3.986013 \times 10^{20} \times (10 \times 10^6) / 6578 \times 10^5$

$= -.6059612344 \times 10^{19}$ ergs

$-GMm_2/r_2 = -3.986013 \times 10^{20} \times (10 \times 10^6) / 19378 \times 10^5$

$= -.2056978532 \times 10^{19}$ ergs

The total energy of each mass is

$kE_1 + PE_1 = .8563195015 \times 10^{18} - .6059612344 \times 10^{19}$

$= -.5203292843 \times 10^{19}$ ergs

$KE_2 + PE_2 = .7431320625 \times 10^{19} - .2056978532 \times 10^{19}$

$= .5374342093 \times 10^{19}$ ergs

The total energy of the system is

$TE_1 + TE_2 = -.5203292843 \times 10^{19} + .5374342093 \times 10^{19}$

$= +.17104925 \times 10^{18}$ ergs

The total angular momentum of the deployed system is

$m(r_1^2 + r_2^2)\omega$

$= 10 \times 10^6 \times [(6578 \times 10^5)^2 + (19378 \times 10^5)^2] \times .6291280459 \times 10^{-3}$

$= 2.634643355 \times 10^{22}$

The angular momentum of a single mass of 20 metric tons at 37,157.64333 km with angular velocity $.6950273674 \times 10^{-4}$ radians/sec is

$2mR^2\Omega = 20 \times 10^6 \times (43535.64333 \times 10^5)^2 \times .6950273674 \times 10^{-4}$

$= 2.634643356 \times 10^{22}$.

The centrifugal force on the mass is

$$2mR\Omega^2 = 20 \times 10^6 \times (43535.64333 \times 10^5) \times (.6950273674 \times 10^{-4})^2$$

$$= 4.206092056 \times 10^8 \text{ dynes,}$$

and the gravitational force is

$$GM(2m)/R^2 = 3.986013 \times 10^{20} \times 20 \times 10^6/(43535.64333 \times 10^5)^2$$

$$= 4.206092054 \times 10^8 \text{ dynes.}$$

The mass is therefore in equilibrium in a circular orbit.

The total energy of the 20 ton mass is

$$2mR\Omega^2/2 - GM(2m)/R$$

$$= 20 \times 10^6 \times (43535.64333 \times 10^5)^2 \times (.6950273674 \times 10^{-4})^2/2$$

$$- 3.986013 \times 10^{20} \times (20 \times 10^6)/43535.64333 \times 10^5$$

$$= 9.155746175 \times 10^{17} - 1.831149235 \times 10^{18}$$

$$= -9.155746175 \times 10^{17} \text{ ergs}$$

and the system is in a bound orbit.

Since the retrieval state has negative total energy and the deployed state has positive energy, the work done during retrieval is

$$\text{WORK} = -9.155746175 \times 10^{17} - .17104925 \times 10^{18}$$

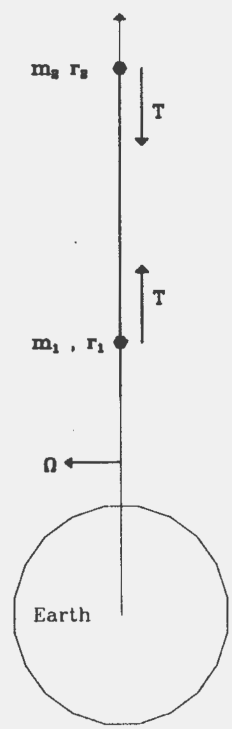
$$= -1.086623868 \times 10^{18} \text{ ergs}$$

which is negative.

9.0 APPENDIX II: ENERGY OF TSS CONFIGURATIONS

by Gordon E. Gullahorn

In this Appendix we derive expressions for the total energy of a dumbbell configuration in circular equilibrium (but not necessarily stable) orbit. (We don't include the effect of tether mass.) These are evaluated and plotted. The physical situation at some chosen time is:



Impose x-y axes at the center of the Earth, in the standard orientation so that the masses lie along the y-axis. We wish the dumbbell to maintain its vertical (relative to Earth) configuration; i.e., the two masses must orbit at the same angular velocity Ω. Thus, the sum of the forces on each mass must equal the acceleration required to keep it in orbit times the mass. The two forces on each mass are the tension T and the gravitational force. The two equations (one for each mass) allow us to determine the two unknowns, Ω and T.

From simple Newtonian mechanics, the forces on each mass are:

$$\vec{F}_1 = (0, -(m_1GM/r_1^2) + T)$$

$$\vec{F}_2 = (0, -(m_2GM/r_2^2) - T)$$

On the other hand, by simple kinematics, the position of the inner mass (aligned as shown at t = 0) is

$$\vec{r}_1 = r_1(\sin \Omega t, \cos \Omega t)$$

and its acceleration

$$\vec{a}_1 = -\Omega^2 r_1(\sin \Omega t, \cos \Omega t)$$

Thus the acceleration at time zero is

$$\vec{a}_1 = (0, -\Omega^2 r_1)$$

and similarly for the outer mass

$$\vec{a}_2 = (0, -\Omega^2 r_2).$$

Balancing the forces with accelerations leads to a set of two equations:

$$-\Omega^2 r_1 m_1 = -(m_1GM/r_1^2) + T$$

$$-\Omega^2 r_2 m_2 = -(m_2GM/r_2^2) - T$$

Before writing down the solution express it in terms of the mass and radius ratios to allow easy scaling:

$$r_2 = r \qquad r_1 = \xi r \qquad 0 < \xi < 1$$

$$m_2 = m \qquad m_1 = \eta m.$$

The restriction on ξ is essentially a naming convention: we shall call the outer mass "mass 2." The mass ratio, η, can take on any positive value depending on which mass is heavier.

Then, neglecting T which we do not need,

$$\Omega^2 = \frac{GM}{r^3}\left[\frac{\eta+\xi^2}{\xi^2(1+\xi\eta)}\right]$$

The energy for each mass, that is the sum of kinetic and gravitational potential energies based on position and velocity and ignoring the tether, is

$$E_1 = -\frac{GM}{r}\left[\frac{\eta}{\xi} - \frac{1}{2}\,\frac{\eta(\eta+\xi^2)}{1+\xi\eta}\right]$$

$$E_2 = -\frac{GM}{r}\left[1 - \frac{1}{2}\,\frac{\eta+\xi^2}{\xi^2(1+\xi\eta)}\right]$$

A program has been written to evaluate these energies and also the total energy $E = E_1 + E_2$. They are then normalized by the (absolute value of the) energy the mass pair would have if in the same orbit:

$$\frac{GMm}{2r}(1+\eta).$$

In Figure 1(a), (b), (c) we plot the normalized energies vs. the ratio of radii, $\xi = r_1/r_2$, each plot for a different mass ratio $\eta = m_1/m_2$. In each case, the top curve (at the left of the plot) is the energy of mass 2, the outer mass; the middle curve is the total energy; and the lower curve is the energy of mass 1, the inner mass. Note that the energy of mass 2 becomes positive for long tethers (small radius ratios), and eventually brings the total energy along with it. At least at a naive level, what is happening is that the outer mass is being forced to traverse its orbit more rapidly than it would without the tether; its kinetic energy increases beyond the normal "orbital" kinetic energy, eventually totally swamping the gravitational potential energy. We have also computed, for each mass ratio η, the values of ξ at which E and E_2 are zero; these are plotted in Figure 2. Note the logarithmic scale on the abscissa. The top curve shows the tether length which zeroes the energy of the outer mass alone, while the lower curve is for the total energy. One interesting fact: the curve for total energy is symmetric about a vertical line through $\log(\xi) = 0$. That is, if we switch the masses in a system, it will take the same tether length (assuming the same outer orbit radius) to reach the transition between negative (bound) and positive total energy; this is not to say that the total energy will be the same for other tether lengths, a matter which we haven't explored. Also note that the equal mass case requires the shortest tether.

In summary, we have confirmed that positive energies are possible in a dumbbell system with sufficiently long tether. The outer mass orbits faster than in a free orbit, in order to maintain a fixed configuration with the inner mass, and its (positive) kinetic energy dominated when the tether connecting it to the inner mass becomes very long. For equal masses, the "optimal" case, the tether must be longer than about 65% of the radius of the outer mass's orbit. The tether length required for positive energy (that is, its ratio to the outer mass orbit radius) depends only on the ratio of the masses, and not on which one is larger.

Appendix II

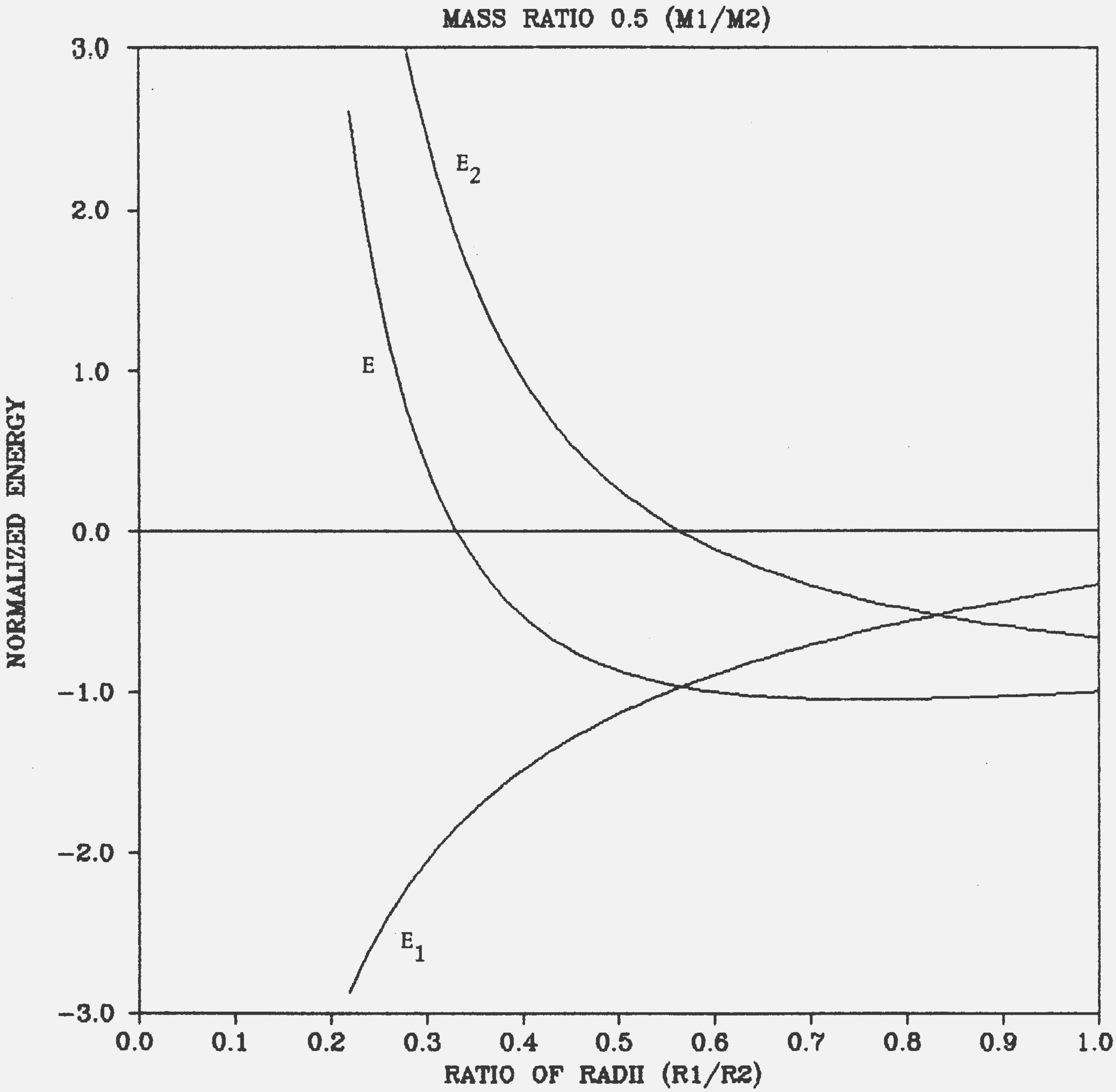


Figure 1a
See text for discussion.

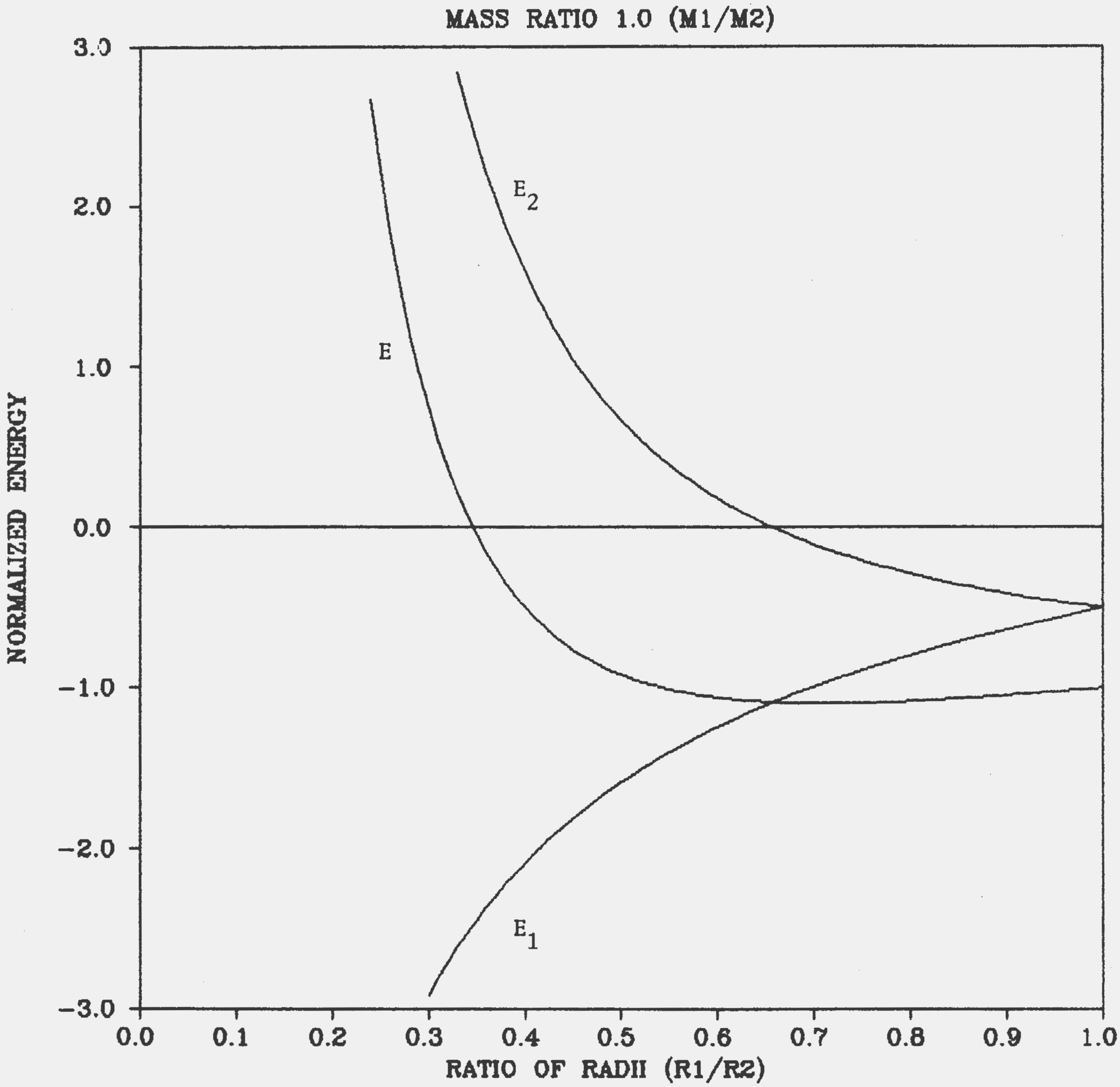

MASS RATIO 1.0 (M1/M2)
NORMALIZED ENERGY
RATIO OF RADII (R1/R2)
3.0
2.0
1.0
0.0
−1.0
−2.0
−3.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
E
E2
E1

Figure 1b

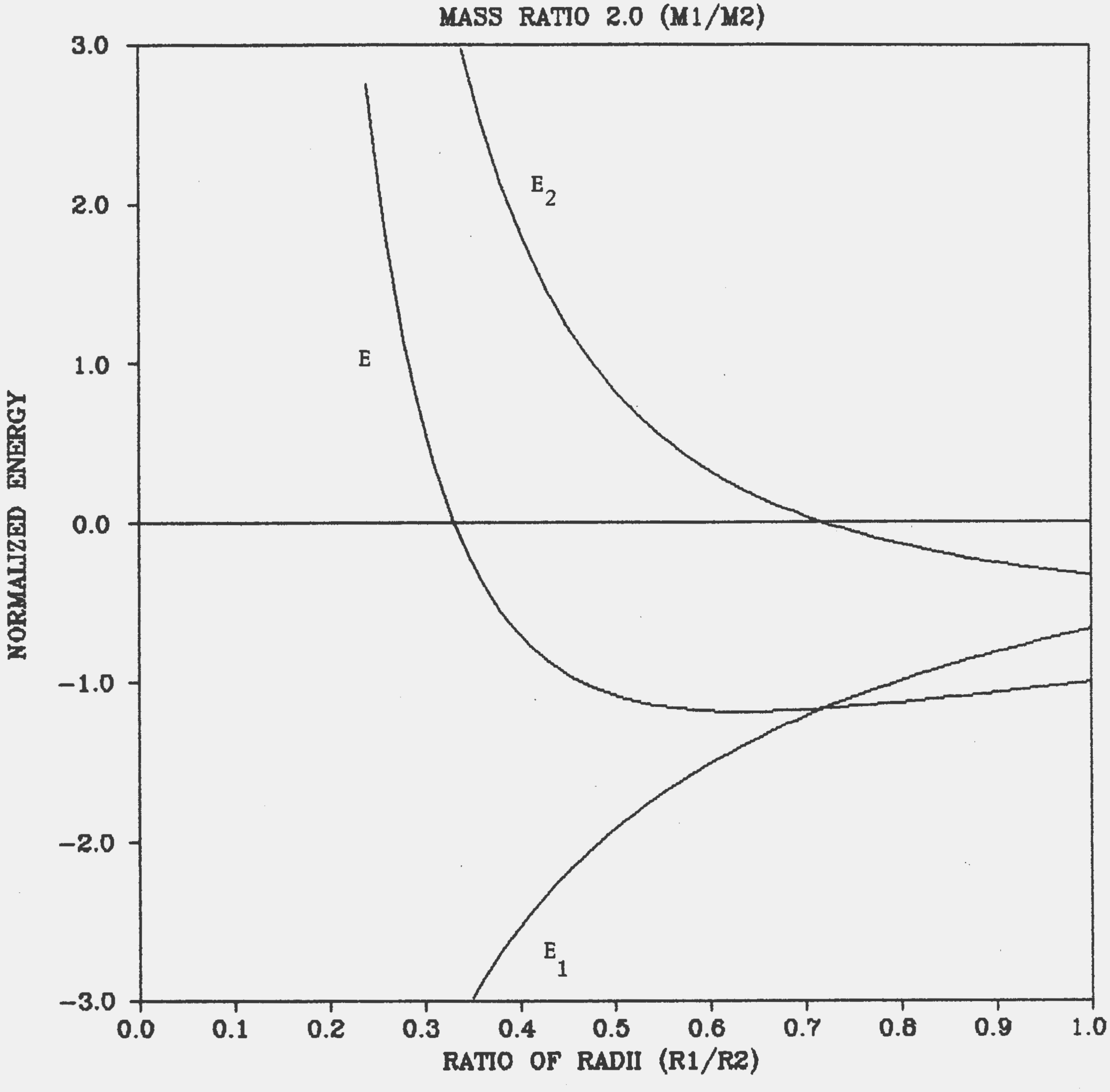

MASS RATIO 2.0 (M1/M2)
3.0
2.0
1.0
0.0
−1.0
−2.0
−3.0
NORMALIZED ENERGY
E
E_2
E_1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
RATIO OF RADII (R1/R2)

Figure 1c

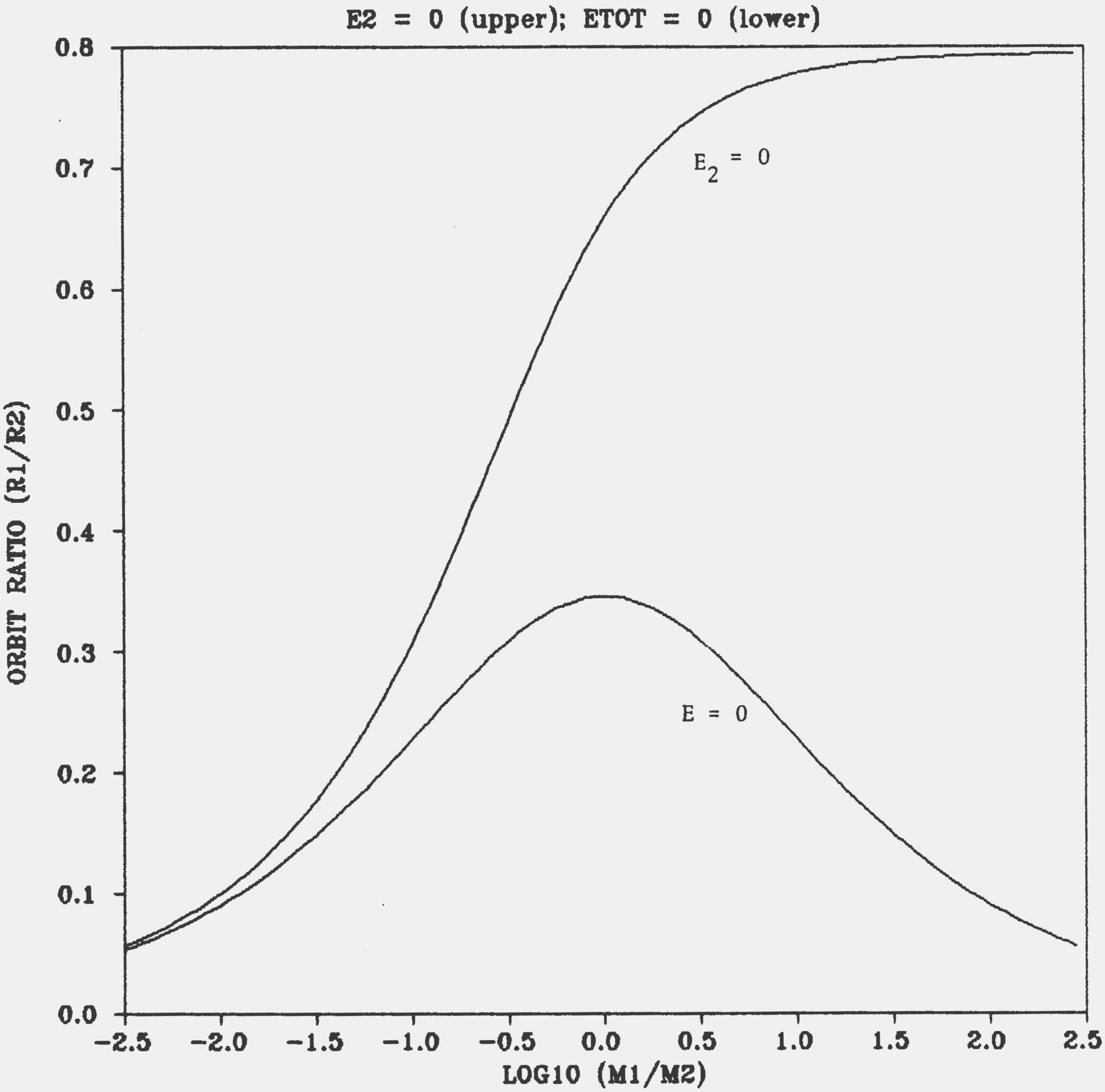


Figure 2

See text for discussion.